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A Stability Analysis on Beck's Procedure for Inverse Heat Conduction Problems

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The sequential function specification method proposed first by Beck is considered as one of the most efficient methods for the inverse heat conduction problem (IHCP) which is extremely ill-posed and time-dependent. This method determines an "inverse solution" advancing in a sequential fashion in time. The values estimated at any given time depend on the solution obtained previously. The main question connected with this method is the stability; *i.e.*, the cumulative error in the solution must remain bounded at all time. Since the first paper of Beck in 1970, few theoretical stability analyses have been studied in the literature. The aim of this paper is to find the conditions under which this method is stable irrespective of the data measurements. For a 1D linear IHCP, we try to construct a sequence

$$X_1 = 1, \qquad X_j = \sum_{l=1}^{j-1} \alpha_{j-l+1} X_l, \quad j \geq 2, \tag{1}$$

such that the coefficients α_i are independent of the data measured and the convergence of the series $\sum_{i=1}^{\infty} |X_i|$ guarantees the stability of the method. In other words, we need to find an adequate condition on α_i such that $\sum_{i=1}^{\infty} |X_i|$ is convergent, implying that the method is stable. The values of α_i depend on the discretization size h of the function to be determined $q(t)$ and the sliding time horizon (or future time interval) τ of the method. The range of values of h and τ which give the values of α_i such that the series $\sum_{i=1}^{\infty} |X_i|$ is convergent is established numerically. Under the stability condition, an error estimation of the Beck's method is derived. The approach presented could be also applied to multidimensional IHCPs, in which the coefficients α_i and X_i are no longer scalar but become square matrices.

1. INTRODUCTION

The determination of surface temperature and/or heat flux from an interior measurement of temperature is referred to as the inverse heat conduction problem and is well known to be ill-posed. To stabilize the numerical inverse solution, various methods have been developed. For example, Tikhonov's regularization is used in [5, 8]. A gradient iterative regularization method is proposed in [1]. A space marching method is suggested in [9]. A mollification method is proposed by Murio [11]. The sequential function specification method or future temperature information method is developed in Beck [2].

In an IHCP (inverse heat conduction problem), where a function $q(t)$ is to be estimated from the observed data $z(t)$, the observed data $z(t)$ lags behind $q(t)$ and is relatively damped. Since Beck's SFSM (sequential function specification method) takes this behavior well into account, this method turns out to be an efficient one for this family of problems. The SFSM is a sequential least-squares method which estimates the unknown time-dependent function one value at a time, in contrast to most other methods which try to obtain all the components simultaneously. The SFSM has been widely applied to different kinds of IHCPs. For example, it has been used in phase change problems to identify the interface position [4] and also to control the interface motion [14]. The SFSM has proved quite efficient and stable, but however, the stability obtained for this method depends on the given data $z(t)$. Different extensions of the method have been developed. In [3] the SFSM is combined with the Tikhonov regularization. In [6] an adaptive method has been developed. Despite the performance of the method in solving IHCPs and the various algorithmic developments it has undergone, to the knowledge of the authors there does not exist a rigorous and precise analysis of the stability of the SFSM. Very recently, some efforts in this direction have been made by Reinhardt [12]. In this paper, we try to analyze the stability of this method with reference to a linear one-dimensional IHCP.

This paper is organized as follows. In Section 2, we give a new formulation for Beck's method which forms the basis for our stability analysis. In Section 3, we derive a stability conditions and an error estimation. In Section 4, we study numerically the stability range of the parameters h and τ. Finally, some theoretical conditions of stability on the coefficients α_i are discussed in the Appendix A.

2. SEQUENTIAL FUNCTION SPECIFICATION METHOD

In this section, we derive a new formulation of Beck's SFSM for a one-dimensional linear IHCP. This formulation is an explicit recursive algorithm and will be used for the stability analysis in the next sections. A 1D linear IHCP with an initial

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condition of uniform zero temperature involves the solution of convolution-type Volterra integral equations of the first kind,

$$\int_0^t \phi'(t-s)q(s)\,ds = z(t), \tag{2}$$

by using Duhamel's theorem, where $q(s)$ is the function to be identified, $z(t)$ is the observation data, and $\phi'(t)$ is the time derivative of the response function $\phi(t)$. The function $\phi(t)$ could be the temperature or the flux on the boundary of observation. It also corresponds to the $z(t)$ when $q(t)$ is a unit step function $\chi(t)$ defined as

$$\chi(t) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0. \end{cases} \tag{3}$$

This problem is ill-posed and is equivalent to the inversion of the convolution mapping

$$\Phi: q(t) \in L^2(0, +\infty) \mapsto \int_0^t \phi'(t-s)q(s)\,ds \in L^2(0, +\infty). \tag{4}$$

The SFSM gives an "inverse solution" to the problem (2) by minimizing a sequential least-squares problems

$$J_i = \int_{(i-1)h}^{(i-1)h+\tau} (\Phi(q)(t) - z(t))^2\,dt \tag{5}$$

for $i = 1, 2, 3, \ldots$, where $h \leq \tau$ is the discretization size of $q(t)$ and the parameter τ is called "sliding time horizon."

For each sequence i, the values of $\Phi(q)(t)$ in the interval $[(i-1)h, (i-1)h + \tau[$ depend not only on the values of $q(t)$ in the same interval $[(i-1)h, (i-1)h + \tau[$ but also on the values of $q(t)$ in the previous interval $[0, (i-1)h[$. For a sought function $q(t)$ discretized using intervals of size h, we have

$$\begin{aligned} \Phi(q)(t + (i-1)h) = \sum_{j=1}^{i=1} \phi_h(t + (i-j)h)q_j \\ + \int_0^t \phi'(t-s)q(s + (i-1)h)\,ds, \end{aligned} \tag{6}$$

with $\phi_h(t) = \phi(t) - \phi(t-h)$ (which is the response subjected to $\chi_h(t) = \chi(t) - \chi(t-h)$). In the original procedure of Beck, the function $q(t)$ was taken to be a constant q_i over the interval $[(i-1)h, (i-1)h + \tau[$ of length τ. The values of $q(t)$ in the interval $[0, (i-1)h[$ are known from the calculations of the preceding sequences $j < i$ and we estimate a specific function $q(t) = q_i$ in the interval $[(i-1)h, (i-1)h + \tau[$ by minimizing

$$\begin{aligned} J_i(q) &= \int_{(i-1)h}^{(i-1)h+\tau} (\Phi(q)(t) - z(t))^2\,dt \\ &= \int_0^\tau (\Phi(q)(t + (i-1)h) - z(t + (i-1)h))^2\,dt. \end{aligned} \tag{7}$$

Then we have the minimizing solution to the problem (5) or (7) of the form

$$q_i = \langle \phi(t), \phi(t) \rangle_{L^2(0,\tau)}^{-1} \langle \phi(t), \hat{z}_i(t) \rangle_{L^2(0,\tau)}, \tag{8}$$

where

$$\hat{z}_i(t) = z(t + (i-1)h) - \sum_{j=1}^{i-1} \phi_h(t + (i-j)h)q_j. \tag{9}$$

The value obtained of q_i is used as the value of $q(t)$ only in the interval $[(i-1)h, ih[$, and the value of $q(t)$ in $[ih, \tau[$ needs to be computed in the next sequence. In a sequence by sequence manner, we obtain the values of $q(t)$ in the all interval $[0, \infty[$. In summary,

THEOREM 2.1. *the Beck's inverse solution to Eq.* (2)

$$B: z(t) \mapsto q(t) = \sum_{i=1}^{\infty} q_i \chi_h(t - (i-1)h) \tag{10}$$

is determined by the algorithm

$$\begin{aligned} q_1 &= \delta_1 \\ q_i &= \delta_i + \sum_{j=1}^{i-1} \alpha_{i-j+1} q_j, \quad i \geq 2, \end{aligned} \tag{11}$$

where the coefficients α_i, δ_i *are determined from*

$$\alpha_i = -\gamma \langle \phi(t), \phi_h(t + (i-1)h) \rangle_{L^2(0,\tau)}, \tag{12}$$

$$\delta_i = \gamma \langle \phi(t), z(t + (i-1)H) \rangle_{L^2(0,\tau)} \tag{13}$$

and the weighting coefficient γ *is defined as*

$$\gamma = \langle \phi(t), \phi(t) \rangle_{L^2(0,\tau)}^{-1} \tag{14}$$

for a given h *and* τ. *Moreover, the above algorithm could be reformulated as*

$$q_i = \sum_{j=1}^{i} X_{i-j+1} \delta_j, \tag{15}$$

where the coefficients X_i *are computed by using*

$$\begin{aligned} X_1 &= 1, \\ X_i &= \sum_{j=1}^{i=1} \alpha_{i-j+1} X_j, \quad i \geq 2. \end{aligned} \tag{16}$$

If the exact sought parameter $\bar{q}(t)$ *is a constant* $\bar{q}_i$ *in each interval* $[(i-1)h, ih[$ *and the observation* $z(t) = \Phi(\bar{q})(t)$, *we have*

$$q_1 - \bar{q}_1 = \varepsilon_1$$
$$q_i - \bar{q}_i = \varepsilon_i + \sum_{j=1}^{i-1} \alpha_{i-j+1}(q_j - \bar{q}_j), \quad i \geq 2, \tag{17}$$

and

$$q_i - \bar{q}_i = \sum_{j=1}^{i} X_{i-j+1\in j},$$

where ε_i *are the errors introduced in each sequence of identification*

$$\varepsilon_i = \gamma < \phi(t), \int_0^t \phi'(t-s)q(s+(i-1)h)\,ds >_{L^2(0,\tau)} - q_i.$$

Proof. Substituting for the function $\hat{z}_i(t)$ from (9) into Eq. (8) and taking (14) into consideration, we obtain the formulae (11).

Now from (11), we deduce Eq. (15) with coefficients X_i determined from (16) by the induction principle. It is easy to check that (15) is true when $i = 1$ from (11). If we assume (15) is true for all $i \leq l$, it remains to show that Eq. (15) is true for $i = l + 1$.

From (11) by setting $i = l + 1$, we have

$$q_{l+1} = \delta_{l+1} + \sum_{j=1}^{l} \alpha_{l+2-j} q_j. \tag{20}$$

Substituting the values of q_j with $j \leq l$ from (15) into the above equation, we obtain

$$\begin{aligned} q_{l+1} &= \delta_{l+1} + \sum_{j=1}^{l} \alpha_{l+2-j} \left(\sum_{k=1}^{j} X_{j-k+1}\delta_k \right) \\ &= \delta_{l+1} + \sum_{j=1}^{l} \sum_{k=1}^{j} \alpha_{l+2-j} X_{j-k+1}\delta_k. \end{aligned} \tag{21}$$

Permuting the index j with k, we have

$$\begin{aligned} q_{l+1} &= \delta_{l+1} + \sum_{k=1}^{l} \sum_{j=k}^{l} \alpha_{l+2-j} X_{j-k+1}\delta_k \\ &= \delta_{l+1} + \sum_{k=1}^{l} \left(\sum_{j=k}^{l} \alpha_{l+2-j} X_{j-k+1} \right) \delta_k, \end{aligned} \tag{22}$$

from which by setting $j = j - k + 1$ we obtain

$$q_{l+1} = \delta_{l+1} + \sum_{k=1}^{l} \left(\sum_{j=1}^{l-k+1} \alpha_{l+3-j-k} X_j \right) \delta_k. \tag{23}$$

Taking into account (16) with $i = l - k + 2$, we finally obtain

$$q_{l+1} = \delta_{l+1} + \sum_{k=1}^{l} (X_{l-k+2})\delta_k = \sum_{k=1}^{l+1} (X_{l-k+2})\,\delta_k. \tag{24}$$

It follows that (15) is true for all $i \geq 1$. ▮

Remark 2.1. Theorem 2.1 is valid for a full discrete IHCP formulated as

$$\sum_{j=1}^{i} a_{i-j+1} q_j = z_i \ \forall\ i \geq 1, \tag{25}$$

which is highly ill-conditioned with $a_1 \approx 0$. For the proof of the Theorem 2.1 in this full discrete situation, it is sufficient to note that $\tau = rh$ with an integer r called the "number of future time steps" and replacing the whole integral by the corresponding summation.

Remark 2.2. Theorem 2.1 can be also extended to a multidimensional IHCP. A multidimensional IHCP discretized in space could be expressed in the form of Eq. (2) as a one-dimensional problem; however, the kernel $\phi(t)$, the sought parameter $q(t)$, and the observation $z(t)$ are not scalar functions of t any more. For each t, $\phi(t)$ becomes an $m \times n$ matrix, $q(t)$ is a vector of length n, and $z(t)$ is a vector of length m. In such a case, the operator $<\cdot,\cdot>_{L^2(0,\tau)}$ should be defined by $< A(t), B(t) >_{L^2(0,\tau)} = \int_0^\tau A^T(t)B(t)\,dt$. Consequently, γ, α_i, X_i become $n \times n$ matrices and δ_i are vectors of length n. $X_1 = 1$ should be replaced by $X_1 = I_{n\times n}$.

3. STABILITY ANALYSIS

In this section, we will study the stability property of the mapping B defined by (11), or by (15) and (16).

THEOREM 3.1. *The Beck inverse mapping B defined by* (11) *is linear and if*

$$E = \sum_{i=1}^{\infty} |X_i| < \infty \tag{26}$$

the mapping B is Lipschitz continuous from $L^\infty(0, +\infty)$ *into* $L^\infty(0, +\infty)$.

Proof. We have shown that Eq. (11) is equivalent to Eq. (15) with the coefficients (16). The coefficients δ_i defined as (13) are linear about the measurement data $z(t)$. The coefficients α_i (12) are independent of the data $z(t)$. Thus X_i determined by (16) are also independent of the data $z(t)$. We see from (15) that the values of solution q_i are linear with respect to δ_i. We have finally that the solution q_i is linear about the data $z(t)$, *i.e.*, the mapping B defined as (10) is linear.

From Eq. (13), we have

$$\begin{aligned} |\delta_i| &= |\gamma|\,|< \phi(t), z(t+(i-1)h) >_{L^2(0,\tau)}| \\ &\leq |\gamma|\,\|\phi(t)\|_{L^1(0,\tau)}\,\|z(t+(i-1)h)\|_{L^\infty(0,\tau)} \\ &\leq |\gamma|\,\|\phi(t)\|_{L^1(0,\tau)}\,\|z(t)\|_{L^\infty(0,\infty)}, \end{aligned} \tag{27}$$

and from (15) we have

$$|q_i| \leq (\max_{1\leq j\leq i} |\delta_j|) \sum_{j=1}^{i} |X_j|. \tag{28}$$

Substituting for $|\delta_i|$ from (27), we obtain

$$\sup_{i\geq 1} |q_i| \leq |\gamma| \, \|\phi(t)\|_{L^1(0,\tau)} \|z(t)\|_{L^\infty(0,\infty)} \left(\sum_{j=1}^{\infty} |X_j| \right). \tag{29}$$

Letting $z(t) = z^1(t) - z^2(t)$ and $q(t) = Bz^1 - Bz^2$ in Eq. (29), we obtain

$$\begin{aligned} \|Bz^1 - Bz^2\|_{L^\infty(0,\tau)} &= \|B(z^1 - z^2)\|_{L^\infty(0,\tau)} = \sup_{i\geq 1} |q_i| \\ &\leq |\gamma| \, \|\phi(t)\|_{L^1(0,\tau)} \left(\sum_{j=1}^{\infty} |X_j| \right) \|z^1(t) - z^2(t)\|_{L^\infty(0,\infty)}. \end{aligned} \tag{30}$$

This means that the mapping B is Lipschitz continuous from $L^\infty(0, +\infty)$ into $L^\infty(0, +\infty)$ as $E < \infty$ and the Lipschitz constant is

$$L = |\gamma| \, \|\phi(t)\|_{L^1(0,\tau)} E. \quad \blacksquare \tag{31}$$

From Eqs. (18) and (19), it is easy to obtain an error estimation as follows.

THEOREM 3.2. *If* $\phi'(t) \geq 0$, $\|\bar{q}'(t)\|_{L^\infty(0,+\infty)} < +\infty$, $\|\Phi(\bar{q})(t) - z(t)\|_{L^\infty(0,+\infty)} \leq \delta$, *and* (26), *we have*

$$\|Bz - \bar{q}\|_{L^\infty(0,+\infty)} \leq C\tau + L\delta, \tag{32}$$

where

$$C = \|\bar{q}'(t)\|_{L^\infty(0,+\infty)} E. \tag{33}$$

Remark 3.1. Theorem 3.1 and Theorem 3.2 are valid for multidimensional IHCPs. The absolute value operator $|\cdot|$ should be interpreted as the norm of a matrix or a vector.

4. THE STABLE REGION OF (h, τ) AND NUMERICAL RESULTS

For a specific IHCP, Beck's solution depends essentially on two parameters: the discretization size h and the sliding time horizon τ. In this section, we study numerically the range of values of the parameters (h, τ) for which the Beck's procedure is stable. For a given value of h and τ, $0 < h \leq \tau$, we can calculate the sequence $\{\alpha_i\}$, and then $\{X_i\}$. Let us define a function E as

$$E(\tau, h) = \sum_{i=1}^{\infty} |X_i|. \tag{34}$$

From Theorem 3.1, we know that the Beck's solution is stable when $E(\tau, h) < +\infty$ and is unstable when $E(\tau, h) = +\infty$. The function could be used as an indication of stability for the Beck's method.

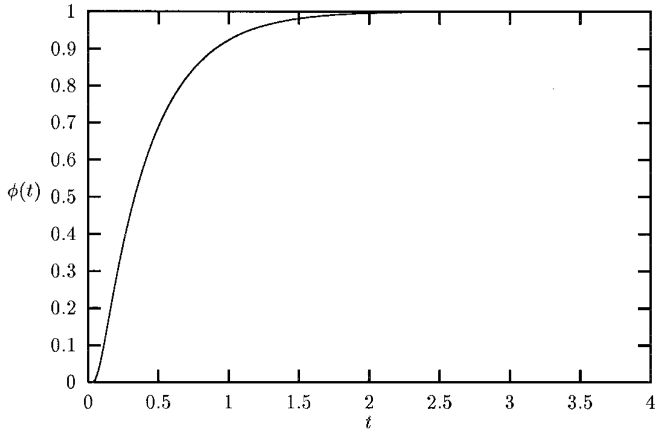

FIG. 1. The response function $\phi(t)$ computed with a discretization $\Delta x = 2^{-4}$ and $\Delta t = 2^{-10}$.

For the numerical calculations, we consider a 1D heat conduction problem in a slab with constant thermal properties. The dimensionless governing equation is

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}, \quad 0 < x < 1, t > 0, \tag{35}$$

with initial condition

$$T(x, 0) = 0. \tag{36}$$

The temperature and flux are assumed known at the location $x = 1$:

$$\frac{\partial T(1, t)}{\partial x} = z(t) \tag{37}$$

$$T(1, t) = 0. \tag{38}$$

It is desired to predict the flux at the location $x = 0$:

$$\frac{\partial T(0, t)}{\partial x} = q(t). \tag{39}$$

The inverse heat conduction problem considered involves determining the heat flux $q(t)$ at $x = 0$ from the given data $z(t)$.

In fact, the mapping $\Phi : q(t) \mapsto z(t) = \partial T(1, t)/\partial x$ is well determined by solving the direct problem represented by Eqs. (35), (36), (38), (39) and then we have Eq. (2). We use simple explicit finite difference method to solve Eqs. (35), (36), (38), (39) with $\Delta t = 2^{-12}$ and $\Delta x = 2^{-4}$. The numerical solution to $\phi(t)$, the heat flux response of a body initially at zero temperature and subjected to a heat flux in the form of a unit step function $\chi(t)$ defined in Eq. (3), is shown in Fig. 1. We can

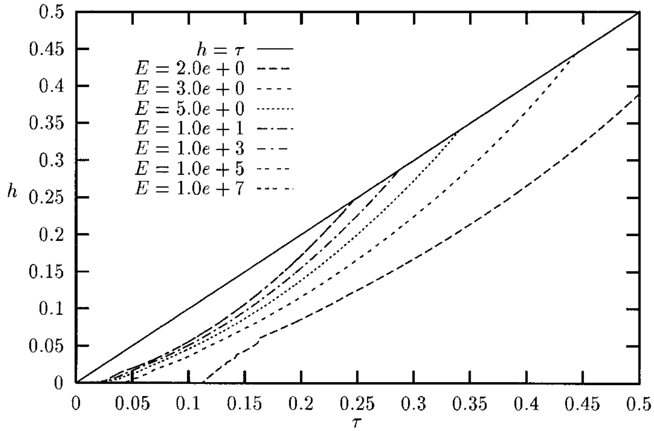

FIG. 2. Contours of $E(\tau, h)$ on $\tau - h$ plane.

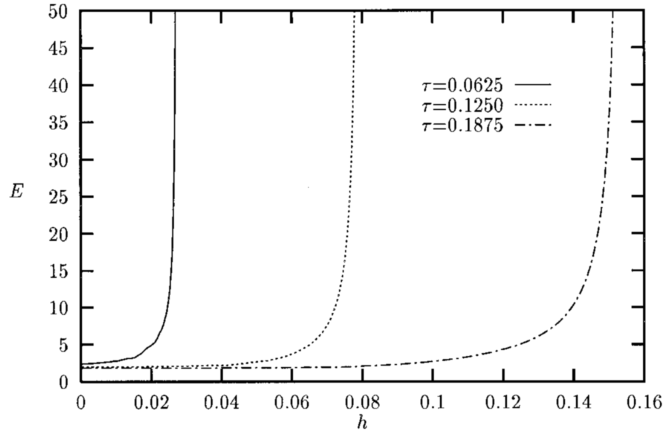

FIG. 4. Stability limit h_{max} at various values of τ.

see from the figure that for small t, $\phi(t)$ is nearly zero. It then increases rapidly and finally attains a constant value with increasing t. Now, we compute numerically the values of $E(\tau, h)$ at all the discrete points $h = m\Delta t$ and $\tau = r\Delta t$ in the domain $0 < h \leq \tau \leq 0.5$ using Eqs. (12), (14), (16), and (34). The function $E(\tau, h)$ is computed for a sample of 2048 values of h and 2048 values of τ in the region $0 < h \leq \tau \leq 0.5$. We consider the domain $\Omega_s = \{(\tau, h)|E(\tau, h) < E_m\}$ as a stable domain and the domain $\Omega_u = \{(\tau, h)|E(\tau, h) \geq E_m\}$ as an unstable domain, where E_m is a numerical representation of the mathematical infinity $(+\infty)$ in Eq. (26). When E_m is sufficiently large, the Ω_s and Ω_u do not vary with E_m. The behavior of $E(\tau, h)$ is very well represented by the contours of constant E in the (τ, h) plane, shown in Fig. 2. It can be seen that the contours of $E = 10^3, 10^5, 10^7$ nearly coincide with one another, indicating that we could choose, for example, $E_m = 10^5$ to represent numerical infinity, in order to identify the stable and unstable domains. Figure 3 presents the domains Ω_s and Ω_u separated by the contour of $E = E_m = 10^5$. These domains Ω_s and Ω_u are simple connected domains.

We can also see from Fig. 3 that the SFSM is unstable for even very small values of h when $\tau \leq 0.01$. The stability behavior for smaller values of τ needs a more detailed investigation; but however, we could remark that smaller values of τ would demand prohibitively small values of h for stability. In the range $0.01 < \tau < 0.247$, we can define an upper limit h_{max} such that $0 < h_{max} < \tau$, where the region $0 < h < h_{max}$ is stable, and $h \geq h_{max}$ is unstable. For $\tau \geq 0.247$, the procedure is stable for all values of h in the region $0 < h \leq \tau$.

Figure 4 presents E, regarded as a function of h, for various values of τ, and Fig. 5 presents the curve of E versus τ for different h values. The function $E(\tau, h)$ is increasing with respect to h and decreasing with respect to τ. From Fig. 4, we see that the method becomes unstable as h increases beyond a certain upper limit h_{max} and that this upper limit h_{max} is a function of τ. At larger τ, the value of h_{max} is higher. Figure 5 presents the converse: for each value of h, there exists a lower limit of τ

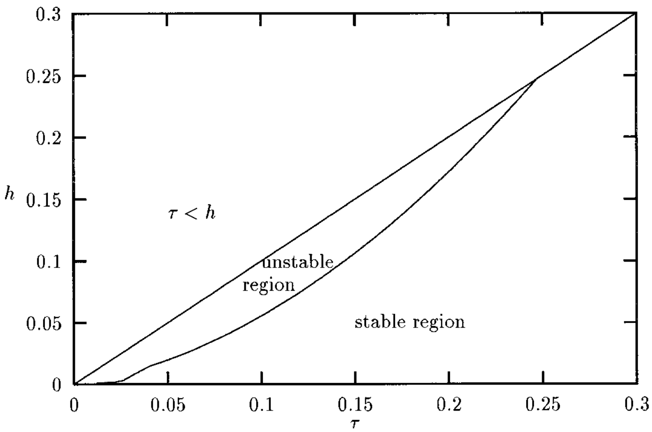

FIG. 3. The stable and unstable range of the parameters (τ, h) obtained numerically.

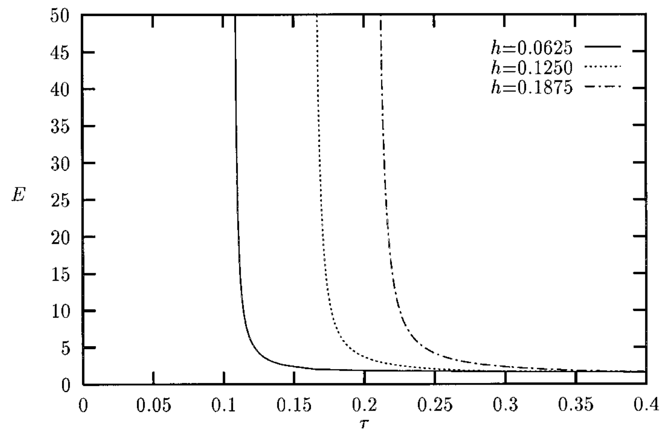

FIG. 5. Stability limit τ_{min} for various values of h.

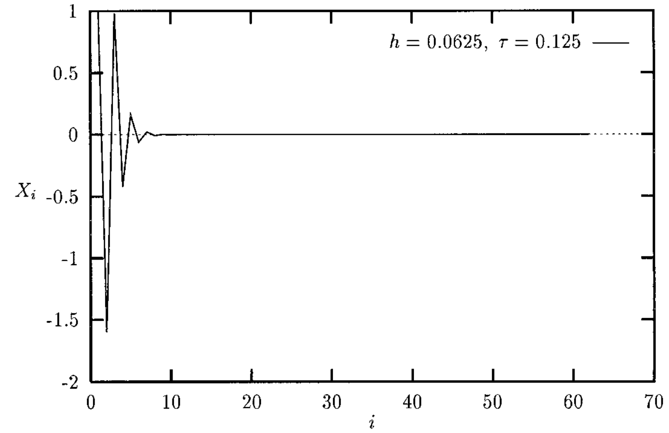

FIG. 6. The behavior of the series $\{X_i\}$ in case 1 ($h = 0.0625$, $\tau = 0.125$, and $E = 4.25$).

($\tau_{\min}$), below which the procedure is unstable. This lower limit $\tau_{\min}$ increases for larger values of h.

The above numerical results are obtained by using the sum of the 1000 terms of the series $\sum_{i=1}^{\infty} |X_i|$ in Eq. (34), *i.e.*, by approximating the $\sum_{i=1}^{\infty} |X_i|$ with $\sum_{i=1}^{1000} |X_i|$. A simple necessary condition for the convergence of $\sum_{i=1}^{\infty} |X_i|$ that $X_i \to 0$ as $i \to \infty$ could be used. To study in more detail the convergent or divergent behavior of the series expansion $\sum_{i=1}^{\infty} |X_i|$, we look at the evolution of the series $\{X_i\}$ and $\{\alpha_i\}$ with respect to the index i of sequence. We choose the following three cases:

1. $h = 0.0625$, $\tau = 0.125$ in the stable region Ω_s where the value of E is relatively small;
2. $h = 0.25$, $\tau = 0.25$ in the stable region Ω_s where the value of E is relatively large;
3. $h = 0.03$, $\tau = 0.06$ in the unstable region Ω_u.

The evolutions of the series $\{X_i\}$ corresponding the three cases are respectively plotted in Figs. 6–8. We see that X_i converges

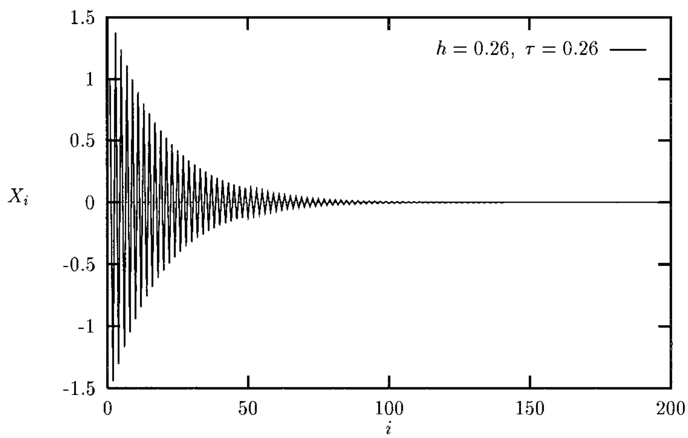

FIG. 7. The behavior of the series $\{X_i\}$ in case 2 ($h = 0.26$, $\tau = 0.26$, and $E = 28.7$).

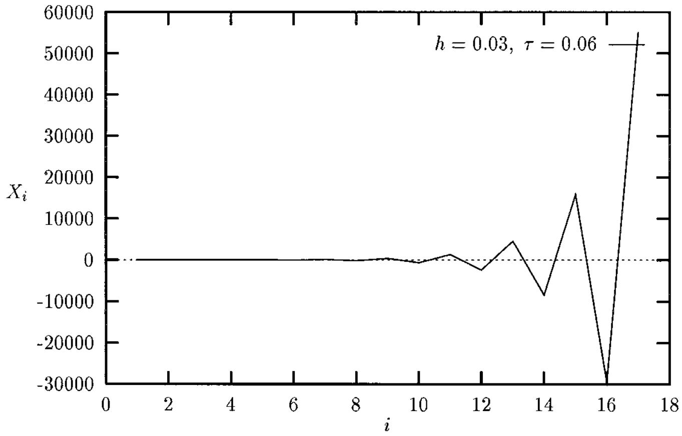

FIG. 8. The behavior of the series $\{X_i\}$ in case 3 ($h = 0.03$, $\tau = 0.06$).

rapidly to zero in case 1; it converges slowly to zero in case 2; it diverges in case 3. A condition such as $|X_i| < \varepsilon$ and/or $|X_i - X_{i-1}| < \varepsilon$ may be used as a stop test of convergence in the computation of the expansion $\sum_{i=1}^{\infty} |X_i|$. We have compared this condition using $\varepsilon = 10^{-3}$ with the method using the sum of 1000 terms of series $\sum_{i=1}^{\infty} |X_i|$. The results obtained are almost identical.

The series $\{\alpha_i\}$ corresponding to the above three cases are shown in Fig. 9. In any case, α_i converges to zero as i tends to ∞. The values of the first few terms of $\{\alpha_i\}$ play an essential role to get a convergent or a divergent series $\{X_i\}$. The values of $\{\alpha_i\}$ in unstable cases have often a larger magnitude than those in stable cases, but however, this is not always true. We remark also that the sufficient conditions of stability given in Theorem A.3 are not optimal, because the $\{\alpha_i\}$ in the above stable cases do not satisfy the conditions imposed by Theorem A.3.

5. CONCLUSIONS

A stability analysis of Beck's sequential function specification method for a linear IHCP has been investigated. Under

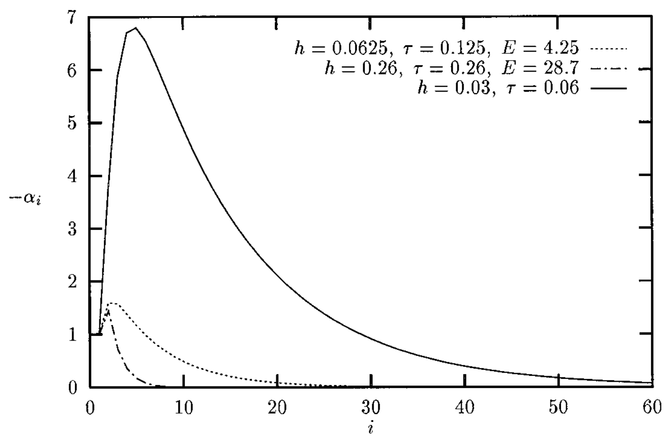

FIG. 9. Comparison of the values of the series $\{\alpha_i\}$ for the three test cases.

the stability condition, an error estimation has been obtained. The present approach could be applied to multidimensional IHCPs. For a 1D case, the decomposition of the parameter domain (h, τ) into stable and unstable regions is precisely computed numerically and some theoretical conditions on the sequence of $\{\alpha_i\}$ have been deduced. More detailed investigation about multidimensional IHCPs needs to be developed. We hope that this study would give a new insight into the Beck's procedure.

APPENDIX A: SOME STABILITY CONDITIONS ON THE COEFFICIENTS $\{\alpha_i\}$

Since the coefficients $\{\alpha_i\}$ are easy to compute directly from the kernel function $\phi(t)$, we are interested in analyzing the property of these coefficients that ensures the stability of Beck's algorithm. Here we limit ourselves to one-dimensional problems (*i.e.*, $\phi(t)$ is a scalar for each fixed t). For multidimensional problems, more detailed investigation needs to be performed when the kernel function $\phi(t)$ is not a simple diagonal matrix. The relation between $\{X_i\}$ and $\{\alpha_i\}$ is not very simple and here are the first five terms of X_i:

$$\begin{aligned} X_1 &= 1, \\ X_2 &= \alpha_2, \\ X_3 &= \alpha_3 + \alpha_2^2, \\ X_4 &= \alpha_4 + \alpha_2\alpha_3 + (\alpha_3 + \alpha_2^2)\,\alpha_2, \\ X_5 &= \alpha_5 + \alpha_2\alpha_4 + (\alpha_3 + \alpha_2^2)\,\alpha_3 \\ &\quad + (\alpha_4 + \alpha_2\alpha_3 + (\alpha_3 + \alpha_2^2)\,\alpha_2)\,\alpha_2. \end{aligned} \tag{40}$$

We will construct two power series which seems more complicated than the original one; however, the advantage is that many theories and algorithms that are well-developed for power series could be applied.

THEOREM A.1. *Let us construct two complex power series functions with the coeffcients* $-\alpha_i$ *and* X_i:

$$\Lambda(z) = 1 - \sum_{i=2}^{\infty} \alpha_i z^{i-1} \tag{41}$$

and

$$\Gamma(z) = \sum_{i=1}^{\infty} X_i z^{i-1}; \tag{42}$$

we have formally that

$$\Lambda(z)\Gamma(z) = 1. \tag{43}$$

A necessary condition for the convergence of the series (26) *is that the series* $\Lambda(z)$ *does not converge to* 0 *in the closed domain* $|z| \le 1$. *A sufficient condition for the convergence of the series* (26) *is that the series* $\Lambda(z)$ (41) *has a convergence radius* $\rho > 1$ *and the function* $\Lambda(z)$ *has no roots in* $|z| \le 1$.

Proof. It is easy to verify that

$$\begin{aligned} (1 - \Lambda(z))\Gamma(z) &= \left(\sum_{i=2}^{\infty} \alpha_i z^{i-1}\right)\left(\sum_{i=1}^{\infty} X_i z^{i-1}\right) \\ &= \sum_{i=1}^{\infty}\left(\sum_{j=1}^{i-1} \alpha_{i-j+1} X_j\right) z^{i-1} = \sum_{i=2}^{\infty} X_i z^{i-1} = \Gamma(z) - 1. \end{aligned} \tag{44}$$

Suppose that the series (26) converges, then the series $\Gamma(z)$ is convergent uniformly and absolutely in the closed domain $|z| \le 1$, since for each term of the series $\Gamma(z)$ we have

$$|X_i z^{i-1}| \le |X_i| \tag{45}$$

for any $|z| \le 1$. Consequently, $|\Gamma(z)| < \infty$ in the all domain $|z| \le 1$. Since $\Gamma(z)\Lambda(z) = 1$, $\Lambda(z)$ has no root in all domain $|z| \le 1$.

Suppose now that the series $\Lambda(z)$ has a convergence radius $\rho > 1$, it is obvious that the function $\Lambda(z)$ is continuous in $|z| < \rho$. Thus, if $\Lambda(z)$ has no roots in $|z| \le 1$, we can find an $\varepsilon > 0$ such that $\Lambda(z)$ has no roots in $|z| < 1 + \varepsilon$. Therefore, $\Lambda(z)$ is analytic and has no roots in $|z| < 1 + \varepsilon$, and the function $\Gamma(z) = 1/\Lambda(z)$ is also an analytic function in $|z| < 1 + \varepsilon$. It follows that the series $\sum_{i=1}^{\infty} X_i z^{i-1}$ converges absolutely in the region $|z| < 1 + \varepsilon$. Letting $z = 1$, we obtain that $\sum_{i=1}^{\infty} |X_i| < +\infty$. ∎

For the sufficient condition, the hypothesis of a convergence radius $\rho > 1$ is expected to be avoided. Let us now consider some particular cases.

1. $\alpha_2 = \alpha$ and $\alpha_i = 0$ $\forall i \ge 3$. In this case, $\Lambda(z) = 1 - \alpha z$ and $X_i = \alpha^{i-1}$ $\forall i \ge 2$. The necessary and sufficient condition for (26) is that $|\alpha| < 1$, which is equivalent to that the function $\Lambda(z)$ has no root in $|z| \le 1$.

2. $\alpha_i = \alpha$, $\forall i \ge 2$. In this case, $\Lambda(z) = 1 - \alpha z/(1 - z)$ in $|z| < 1$ and $X_i = \alpha(1 + \alpha)^{i-2}$, $\forall i \ge 2$. The necessary and sufficient condition for (26) is that $|1 + \alpha| < 1$. Indeed, this is also that the function $\Lambda(z)$ has no root in $|z| \le 1$.

3. $\alpha_i = 2(-1)^{i-1}$, $\forall i \ge 2$. We have that $\Lambda(z) = (1 - z)/(1 + z)$ has no roots in $|z| < 1$ but has a root $z = 1$ in $|z| \le 1$. In this case, the values of X_i computed from (16) is equal to 2 for all $i \ge 2$ and the series $\sum_{i=1}^{\infty} |X_i|$ does not converge.

When the series $\Lambda(z)$ has only few terms, the condition that $\Lambda(z)$ has no roots in $|z| \le 1$ could be easily checked. In general the series $\Lambda(z)$ has many terms or infinite terms, the condition of $\min_{|z|\le 1} \Lambda(z) \ge \varepsilon > 0$ could be verified by using the Schur's algorithm [13, 7, 10]. However, the computation may not be as fast as to compute directly the coefficients $\{X_i\}$. A comparative study is needed.

We are now interested in some simple but more restrictive sufficient conditions for stability. Indeed, it is easy to proof:

LEMMA A.2. *The values of the sequence* $\{\alpha_i\}$ *derived from* (12) *are negative and the series* $\sum_{i=2}^{\infty} |\alpha_i|$ *is convergent, under the condition that the response function* $\phi(t)$ *is increasing and bounded.*

THEOREM A.3. *Suppose that the power series* (41) *has a convergence radius* $\rho > 1$ *and that* $\alpha_i \leq 0$. *Then, we have the sufficient conditions for the convergence of the series* (26) *as*

$$\sum_{i=2}^{\infty} |\alpha_i| < 1; \tag{46}$$

or

$$\sum_{i=2}^{\infty} |\alpha_i| = 1, \quad \alpha_2 \neq 0, \alpha_3 \neq 0; \tag{47}$$

or

$$-1 < \alpha_i \leq \alpha_{i+1} \leq 0 \quad \forall i \geq 2. \tag{48}$$

Proof. On the basis of Theorem A.1, it is sufficient to check $|\Lambda(z)| > 0$ in the closed region $|z| \leq 1$. Let $|z| \leq 1$, then

$$\left| \Lambda(z) \right| \geq \left| 1 - \sum_{i=2}^{\infty} |\alpha_i| |z|^{i-1} \right| . \tag{49}$$

Therefore, $|\Lambda(z)| > 0$ when the condition (46) holds.

Let condition (47) hold. Suppose that $\Lambda(z)$ has a root z_0 in $|z| \leq 1$; we have

$$z_0 \sum_{i=2}^{\infty} \alpha_i z_0^{i-2} = 1. \tag{50}$$

On the other hand,

$$\left| z_0 \sum_{i=2}^{\infty} \alpha_i z_0^{i-2} \right| \leq |z_0| \sum_{i=2}^{\infty} |\alpha_i z_0^{i-2}| \leq \sum_{i=2}^{\infty} |\alpha_i| \leq 1. \tag{51}$$

Then $|z_0| = 1$ and

$$\left| \sum_{i=2}^{\infty} \alpha_i z_0^{i-2} \right| = \sum_{i=2}^{\infty} |\alpha_i z_0^{i-2}|. \tag{52}$$

Thus, the nonzero values of the complex numbers $\alpha_i z_0^{i-2}$ must have the same angle. Since $\alpha_2 \neq 0$ and $\alpha_3 \neq 0$, we obtain that $z_0 = 1$. However, $\Lambda(1) \geq 1$ means that $\Lambda(z)$ has no roots in $|z| \leq 1$.

Suppose now that the condition (48) holds. Let us look at the power series function $(1 - z)\Lambda(z)$. The function has the coefficients $\beta_i = \alpha_{i-1} - \alpha_i \leq 0$ with $\alpha_1 = -1$. It is easy to check that $\sum_{i=1}^{\infty} |\beta_i| = 1$. In the same ways as for the condition (47), we can show that the $(1 - z)\Lambda(z)$ has only root $z = 1$. Note that $\Lambda(1) \geq 1$; it follows that $\Lambda(z)$ has no roots in $|z| \leq 1$. ∎

The following examples show that the conditions $\alpha_2 \neq 0$, $\alpha_3 \neq 0$ in (47) and $\alpha_i \neq -1$ in (48) are necessary.

EXAMPLE 1. Let $\alpha_2 = -1/2$, $\alpha_3 = 0$, $\alpha_4 = -1/2$, and $\alpha_i = 0$ $\forall i \geq 5$; we have that $\sum_{i=2}^{\infty} |\alpha_i| = 1$, but, however, the function

$$\Lambda(z) = 1 + z/2 + z^3/2 \tag{53}$$

has a root $z = -1$.

EXAMPLE 2. Let $\alpha_2 = -1$, $\alpha_3 = -1/2$, $\alpha_4 = -1/2$, and $\alpha_i = 0$ $\forall i \geq 5$; we have that the sequence α_i is increasing; however, function

$$\Lambda(z) = 1 + z + z^2/2 + z^3/2 \tag{54}$$

has a root $z = -1$.

APPENDIX B: NOMENCLATURE

$q(t)$:	unknown function to be estimated
$z(t)$:	observed measurement data
$\phi(t)$:	response function
h:	time discretization step for $q(t)$
τ:	length of sliding time horizon
$E(\tau, h)$:	indication function of stability
γ:	weighting value determined by $\phi(t)$
α_i:	sequence define by eq. (12)
δ_i:	weighted observation data
X_i:	sequence define by eq. (16)
$h_{\max}$:	upper bound of h for stability
$\tau_{\min}$:	lower bound of τ for stability
z:	complex variable
$\Gamma(z)$:	power series defined by (42)
$\Lambda(z)$:	power series defined by (41)
r:	ratio τ/h, i.e., number of future informations
E_m:	numerical infinity
i:	index of sequence in the SFSMs
Φ:	direct mapping to be inversed
$\chi(t)$:	unit step function
B:	Beck inverse solution mapping
Ω_s:	stable domain
Ω_u:	unstable domain.

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